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P-ADICALLY F-PURE TYPE SINGULARITY
(AND QUASI P-ADICALLY F-PURE SINGULARITY)

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Introduction

Hara-Watanabe defined F-pure singularity of a pair (R, I^t) , where R is a ring of characteristic $p > 0$, I an ideal of R and t a positive rational number. It seemed a positive characteristic analogue of log canonical singularity of a pair. In fact, for a ring R_0 of characteristic zero, an ideal I_0 and $t \in \mathbb{Q}_{>0}$, if a pair (R_0, I_0^t) is F-pure after reduction to p for infinitely many prime p , then the pair (R_0, I_0^t) should be log canonical! **However, $(R, I^{\text{fpt}(I)})$ is not necessary F-pure, although $(R_0, I_0^{\text{lct}(I_0)})$ is always log canonical. The author wants to generalize F-pure singularity to satisfy that $(R, I^{\text{fpt}(I)})$ is always F-pure.**

$F : R \rightarrow F_*R$: Frobenius map

Def 1 (Hara-Watanabe) (R, I^t) is F-pure if $\forall e \gg 0$, $\exists f \in I^{\lfloor t(p^e-1) \rfloor}$ such that

$$fF^e : R \rightarrow F_*^e R (x \mapsto fx^{p^e})$$

splits as an R -hom.

Def 2 (Takagi-Watanabe) F-pure thresholds are def. as

$$\begin{aligned} \text{fpt}(I) &:= \sup\{t \in \mathbb{Q} \mid (R, I^t) \text{ is F-pure}\} \\ &= \inf\{t \in \mathbb{Q} \mid (R, I^t) \text{ is not F-pure.}\} \end{aligned}$$

We can compute F-purity and $\text{fpt}(I)$ by the Following Fedder's type criterion: Suppose that (R, \mathfrak{m}) is regular local. For $f \in R$, $e \in \mathbb{N}$,

$$fF^e : R \rightarrow F_*^e R$$

splits iff

$$f \in \mathfrak{m}^{[p^e]} := (g^{p^e} \mid g \in \mathfrak{m}).$$

Ex 1 (F-pure thresholds)

$$I := (x^2 + y^3) \subseteq R := k[x, y]$$

p	$\text{fpt}(I)$	
2	1/2	
3	2/3	
$p \equiv 5 \pmod{6}$	$(5p-1)/6p$	
$p \equiv 1 \pmod{6}$	5/6	$= \text{lct}(x^2 + y^3)$

For $I_0 \subseteq R_0$:an ideal of f.g. \mathbb{C} -algebra R_0 , say $I_p \subseteq R_p$ is its reduction to char. p . (R_0, I_0^t) is F-pure type if \exists infinitely many prime numbers p such that (R_p, I_p^t) is F-pure.

Thm 1 (Hara-Watanabe)
Assume that R_0 : \mathbb{Q} -Gorenstein, normal over \mathbb{C} .
 (R_0, I_0^t) : F-pure type \Rightarrow log canonical.

Hara proved $(R, (f)^{\text{fpt}(f)})$:F-pure, (f) : principal.

Ex 2 $((R, I^{\text{fpt}(I)})$: not noc. F-pure)

$$I := (x^2, y^2, z^2) \subseteq R := k[x, y, z]$$

Then $\forall p > 0, \text{fpt}(I) = \frac{3}{2} (= \text{lct}(I))$

$p = 2$	$(R, I^{\text{fpt}(I)})$: NOT F-pure
p : otherwise	$(R, I^{\text{fpt}(I)})$: F-pure

$\nu(t) := -\min\{0, \nu_p(t)\}$, $\nu_p(-)$ is p-adic valuation
Def 3 (H) (R, I^t) is **p-adically F-pure** if \exists infinitely many e with $p^{\nu(t)}t(p^e-1) \in \mathbb{Z}$, $\exists f \in I^{p^{\nu(t)}t(p^e-1)}$ such that

$$fF^{e+\nu(t)} : R \rightarrow F_*^{e+\nu(t)} R (x \mapsto fx^{p^{e+\nu(t)}})$$

splits as an R -hom.

Rem 1 We don't need round down $\lfloor \rfloor$!
(We don't kill information of an exponent t .)

Prop 1 (H)

$$\text{fpt}(I) = \inf\{t \in \mathbb{Q} \mid (R, I^t) \text{ is not p-adic. F-pure.}\}$$

Prop 2 (H)

$$\text{F-pure} \Rightarrow \text{p-adic. F-pure.}$$

Ex 3 I, R : same as in Ex2. Then $\forall p > 0$,

$$(R, I^{\text{fpt}(I)}) : \text{p-adic. F-pure.}$$

Thm 2 (H) Suppose that $I = (f)$ is principal.
 $(R, (f)^{t'})$ is p-adic. F-pure, $\forall t' \leq t$
 $\Rightarrow (R, (f)^t)$ is F-pure.

p-adic. F-pure type can be defined like F-pure type.

Thm 3 (H) R_0 : \mathbb{Q} -Gorenstein, normal over \mathbb{C} .
 (R_0, I_0^t) : p-adic. F-pure type \Rightarrow log canonical.

(Pf. of thm3.) Use Schwede's sharply F-purity;
 (R, I^t) is sharply F-pure if \exists infinitely many e , $\exists f \in I^{\lceil t(p^e-1) \rceil}$ such that

$$fF^e : R \rightarrow F_*^e R (x \mapsto fx^{p^e})$$

splits as an R -hom.

He proved sharply F-pure type implies log canonical. For fixed t , there is only finite prime numbers p such that $\nu_p(t) < 0$. If $\nu_p(t) \geq 0$, then sharp. F-pure = p-adic. F-pure. Therefore

$$(R_0, I_0^t) : \text{p-adic. F-pure type} \Leftrightarrow \text{sharply F-pure type.}$$

(q.e.d.)

Ex 4 $((R, I^{\text{fpt}(I)})$: not nec. p-adic. F-pure)

$$I := (x^2, y^2) \subseteq R := k[x, y]$$

Then $\forall p > 0$, $\text{fpt}(I) = 1 (= \text{lct}(I))$.

$$\begin{aligned} (R, I^{\text{fpt}(I)}) &: \text{p-adic. F-pure if } p \neq 2, \\ &: \text{quasi p-adic. F-pure } \forall p. \end{aligned}$$

Def 4 (R, I^t) is **quasi p-adically F-pure** if $\exists \nu \geq 0$, \exists infinitely many e with $p^{\nu}t(p^e-1) \in \mathbb{Z}$, $\exists f \in I^{p^{\nu}t(p^e-1)}$ such that

$$fF^{e+\nu} : R \rightarrow F_*^{e+\nu} R (x \mapsto fx^{p^{e+\nu}})$$

splits as an R -hom.

Thm 4 (H) $R := k[x_1, \dots, x_n]$, I its monomial ideal. Then

$$(R, I^{\text{fpt}(I)}) \text{ is quasi p-adic. F-pure.}$$